MA 214 - Problem Set 1 Solutions

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(1) We have the points $(0,0),(0.5,y),(1,3),(2,2)$. We also know that the coefficient of x^3 in the interpolation polynomial $p_3(x)$ is 6. Using divided differences, the coefficient of x^3 is equal to $f[x_0,x_1,x_2,x_3]$. We solve for y as follows:

$$\begin{aligned}
f[x_0] &:= f(x_0) = 0 \\
\therefore f[x_0,x_1] &:= \frac{f(x_1)-f(x_0)}{x_1-x_0} = 2y \\
f[x_1,x_2] &:= \frac{f(x_2)-f(x_1)}{x_2-x_1} = 6-2y \\
f[x_2,x_3] &:= \frac{f(x_3)-f(x_2)}{x_3-x_2} = -1 \\
\therefore f[x_0,x_1,x_2] &:= \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = 6-4y \\
f[x_1,x_2,x_3] &:= \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{4y-14}{3} \\
\therefore f[x_0,x_1,x_2,x_3] &:= \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0} = \frac{16y-32}{6}
\end{aligned}$$

Equating this expression to the coefficient of x^3, we get the value for $y = 4.25$. ■

(2) Since we have only 2 points $x_0=-1, x_1=1$, our interpolation polynomial $p_1(x)$ will be a straight line. Since there is no information about the function values at these points, p_1 will be in terms of $y_0=f(x_0)$ and $y_1=f(x_1)$:

$$\begin{aligned}
p_1(x) &= f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) \\
p_1(x) &= y_0 + \frac{y_1-y_0}{2}(x+1) \\
\therefore p_1(x) &= \frac{y_1+y_0}{2} + \left(\frac{y_1-y_0}{2}\right)x
\end{aligned}$$

In order to get the required inequality, we use the error equation as follows:

$$\begin{aligned}
\max_{x\in[a,b]}|f(x)-p(x)| &\leqslant \frac{1}{(n+1)!}||f^{(n+1)}|| \max_{x\in[a,b]}\prod_{k=0}^{n}(x-x_k) \\
\therefore \max_{x\in[-1,1]}|f(x)-p_1(x)| &\leqslant \frac{1}{2}||f''|| \max_{x\in[-1,-1]}\prod_{k=0}^{1}(x-x_k) \\
\therefore \max_{x\in[-1,1]}|f(x)-p_1(x)| &\leqslant \frac{1}{2}\max_{x\in[-1,1]}|f''(x)| \max_{x\in[-1,-1]}(x^2-1)
\end{aligned}$$

We observe that $\max\limits_{x\in[-1,-1]}(x^2-1)$ takes the value of 1 (namely at $x=0$), which gives us the desired inequality. ■

(3) We have the function $f(x) = \sqrt{x - x^2}$ and the points $x_0 = 0, x_1 = a, x_2 = 1$. We can solve for $p_2(x)$ using the Lagrange interpolation formula as follows:

$$
\begin{aligned}
L_0^2(x) &= \frac{(x-1)(x-a)}{(0-a)(0-1)} = \frac{x^2-(a+1)x+a}{a} \\
L_1^2(x) &= \frac{(x-1)(x-0)}{(a-0)(a-1)} = \frac{x^2-x}{a^2-a} \\
L_2^2(x) &= \frac{(x-0)(x-a)}{(1-a)(1-0)} = \frac{x^2-ax}{1-a} \\
\therefore p_2(x) &= 0\left(\frac{x^2-(a+1)x+a}{a}\right) + \sqrt{a-a^2}\left(\frac{x^2-x}{a^2-a}\right) + 0\left(\frac{x^2-ax}{1-a}\right) \\
&= -\left(\frac{x^2-x}{\sqrt{a-a^2}}\right)
\end{aligned}
$$

Thus the required value of a is ■

(4) We can prove the result by simply substituting values of i and taking 3 cases as follows:

i) for $i = 0$:

$$p_{n+1}(x_0) = \frac{(x_0-x_0)r_n(x_0)-(x_0-x_{n+1})q_n(x_0)}{x_{n+1}-x_0} = \frac{0+(x_0-x_{n+1})0}{x_{n+1}-x_0} = 0$$

ii) for $i = 1, 2, \ldots, n$:

$$p_{n+1}(x_i) = \frac{(x_i-x_0)r_n(x_i)-(x_i-x_{n+1})q_n(x_i)}{x_{n+1}-x_0} = \frac{(x_i-x_0)0+(x_i-x_{n+1})0}{x_{n+1}-x_0} = 0$$

iii) for $i = n+1$:

$$p_{n+1}(x_{n+1}) = \frac{(x_{n+1}-x_0)r_n(x_{n+1})-(x_{n+1}-x_{n+1})q_n(x_{n+1})}{x_{n+1}-x_0} = \frac{(x_{n+1}-x_0)0+0}{x_{n+1}-x_0} = 0$$

By uniqueness theorem, $p_{n+1}(x)$ is the only polynomial of degree $n+1$ interpolating on the given points. ■

(5) We are given the interpolation points $x_0 = 0, x_1 = 0.4, x_2 = 0.7$ and the divided differences as $f[x_2] = 6, f[x_1, x_2] = 10, f[x_0, x_1, x_2] = \frac{50}{7}$. The rest of the values can found as follows:

$$
\begin{aligned}
f[x_0, x_1] &= f[x_1, x_2] - f[x_0, x_1, x_2](x_2 - x_0) \\
&= 10 - \left(\frac{50}{7}\right)0.7 = 5 \\
f[x_1] &= f[x_2] - f[x_1, x_2](x_2 - x_1) \\
&= 6 - 10(0.7 - 0.4) = 3 \\
f[x_0] &= f[x_1] - f[x_0, x_1](x_1 - x_0) \\
&= 3 - 5(0.4) = 1
\end{aligned}
$$

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(6) Using the given data, we can calculate the divided differences as follows (intermediate calculations have been skipped):

$$
\begin{aligned}
f[x_0] &= f(x_0) = 1 \\
\therefore f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 3 \\
\therefore f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = 2 \\
\therefore f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = -1 \\
\therefore f[x_0, x_1, x_2, x_3, x_4] &= 0 \\
\therefore f[x_0, x_1, x_2, x_3, x_4, x_5] &= 0
\end{aligned}
$$

Hence $p(x) = 1 + 3(x+2) + 2(x+2)(x+1) - x(x+2)(x+1)$. ■

(7) We have $f \in \mathcal{C}^n[a,b]$ and $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$. Using the divided differences method, the interpolation polynomial can be written as

$$p(x) := f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n] \prod_{k=0}^{n-1} (x - x_k)$$

Since $f(x_i) = p(x_i)$ for $i = 0, 1, \ldots, n$, thus the function $f(x) - p(x)$ has $n+1$ distinct roots. Thus by Rolle's theorem, $f^{(n)}(x) - p^{(n)}(x)$ has exactly one root in $[a,b]$. Let this root occur at $x = \delta$. Thus,

$$f^{(n)}(\delta) - p^{(n)}(\delta) = 0$$

Now, note that the n^{th} order derivative of p is identically $n! f[x_0, x_1, \ldots, x_n]$. ■

(8) The solution is similar to that of question 7 above. ■