MA 214 - Problem Set 2 Solutions

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(1) Assume that $xp(x-1) = (x+1)p(x)$ is true. Take a function $f(x) = (x+1)p(x)$. Then for some real α, we have $f(x-1) = f(x) \implies \cdots = f(-1) = f(0) = f(1) = \cdots = \alpha$ i.e. $f(x) - \alpha$ has infinitely many roots.

Define $g(x) = f(x) - \alpha$. Thus g has infinitely many zeroes. This is a contradiction to our assumption, because no polynomial of finite degree can have infinitely many roots. Therefore, g must identically be zero. We get $p(x) = \dfrac{\alpha}{x+1}$, which can be a polynomial only if $\alpha = 0$. Thus p is identically zero. ■

(2) We can directly compute all these values as follows:

$$\begin{aligned}
d_1 &= \max_{x\in[-\pi,\pi]} |\cos x - \sin x| = \sqrt{2} \\
d_2 &= \max_{x\in[-\pi,\pi]} |\cos^2 x - \sin^2 x| = \max_{x\in[-\pi,\pi]} |1 - 2\sin^2 x| = 1 \\
d_3 &= \max_{x\in[-\pi,\pi]} |\cos^3 x - \sin^3 x| \\
&= \max_{x\in[-\pi,\pi]} |\cos x - \sin x| \max_{x\in[-\pi,\pi]} |1 - \sin x \cos x| \\
&= \max_{x\in[-\pi,\pi]} |\sqrt{2}\cos\left(x + \frac{\pi}{2}\right)| \max_{x\in[-\pi,\pi]} |1 - \frac{\sin 2x}{2}| = 1
\end{aligned}$$

Thus $d_1 > d_2 = d_3$. ■

(3) **I** $\rightarrow \begin{cases} \text{for } a_k x^k \rightarrow k \text{ multiplications, } \therefore \text{ for } k = 0, 1, \ldots, n \implies \text{total } \dfrac{n(n+1)}{2} \\ n \text{ additions} \end{cases}$

II $\rightarrow \begin{cases} \text{for } a_k x^k \rightarrow 2 \text{ multiplications for } k = 2, \ldots, n \text{ and } 1 \text{ for } a_1 x \implies \text{total } 2n-1 \\ n \text{ additions} \end{cases}$

III $\rightarrow$ 1 addition and multiplication in each bracket, $\therefore$ n additions and multiplications ■

(4) We have 3 intervals and want to interpolate on them using quadratic splines. Define the piecewise quadratic as:

$$\varphi = \begin{cases} \varphi_1 = a_0^{(1)} + a_1^{(1)}x + a_2^{(1)}x^2 \text{ on } [-1, 0] \\ \varphi_2 = a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2 \text{ on } [0, 1] \\ \varphi_3 = a_0^{(3)} + a_1^{(3)}x + a_2^{(3)}x^2 \text{ on } [1, 2] \end{cases}$$

Using boundary conditions on φ, we get $a_0^{(1)} - a_1^{(1)} + a_2^{(1)} = 1$ and $a_0^{(3)} + 2a_1^{(3)} + 4a_2^{(3)} = 8$.

Using the continuity of φ at interior points $x = 0$ and $x = 1$, we get

$$\varphi_1(0) = \varphi_2(0) = 0 \implies a_0^{(1)} = a_0^{(2)} = 0$$
$$\varphi_2(1) = \varphi_3(1) \implies a_0^{(2)} + a_1^{(2)} + a_2^{(2)} = a_0^{(3)} + a_1^{(3)} + a_2^{(3)} = 1$$

Finally, we use the condition of continuity of the first derivative at interior points:

$$\varphi_1'(0) = \varphi_2'(0) \implies a_1^{(1)} = a_1^{(2)}$$
$$\varphi_2'(1) = \varphi_3'(1) \implies a_1^{(2)} + 2a_2^{(2)} = a_1^{(3)} + 2a_2^{(3)}$$

The last condition is given to us as $a_2^{(3)} = 0$. We can write this in form of the following matrix equation:

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ a_0^{(2)} \\ a_1^{(2)} \\ a_2^{(2)} \\ a_0^{(3)} \\ a_1^{(3)} \\ a_2^{(3)} \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

■

(5) (a) True

(b) Use the binomial formula

(c) Keeping in mind that $B_{n-1,-1}(x) = 0$,

$$\sum_{k=0}^{n} \frac{k}{n} B_{n,k}(x) = \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x^k) = x \sum_{k=0}^{n} B_{n-1,k-1}(x) = x$$

(d) We expand tha given expression as follows:

$$\begin{aligned} \sum_{k-0}^{n} \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) &= \sum_{k-0}^{n} \frac{k^2}{n^2} B_{n,k}(x) - 2x \sum_{k-0}^{n} \frac{k}{n} B_{n,k}(x) + x^2 \sum_{k-0}^{n} B_{n,k}(x) \\ &= \sum_{k-0}^{n} \frac{k^2}{n^2} \binom{n}{k} x^k (1-x)^{n-k} - x^2 \\ &= \sum_{k-0}^{n} \frac{k}{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} - x^2 \\ &= \sum_{k-0}^{n} \frac{1}{n} \left[(k-1)\binom{n-1}{k-1} + \binom{n-1}{k-1} \right] x^k (1-x)^{n-k} - x^2 \\ &= \sum_{k-0}^{n} \frac{1}{n} \left[(n-1)\binom{n-2}{k-2} + \binom{n-1}{k-1} \right] x^k (1-x)^{n-k} - x^2 \\ &= \frac{(n-1)x^2 + x}{n} - x^2 = \frac{x(1-x)}{n} \end{aligned}$$

(e) We have $B_n(f) - f = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,k}(x) - f(x)$. We can write $f(x) = \sum_{k=0}^{n} B_{n,k}(x) f(x)$.

$$\therefore B_n(f) - f = \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x) \right) B_{n,k}(x)$$

$$\therefore |B_n(f) - f| \leqslant \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x)$$

We now make use of the information that for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $|x - y| < \delta$.

$$\sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) = \sum_{\left|\frac{k}{n} - x\right| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) + \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x)$$

$$S_1 = \sum_{\left|\frac{k}{n} - x\right| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) \leqslant \frac{\varepsilon}{2} \sum_{\left|\frac{k}{n} - x\right| < \delta} B_{n,k}(x) \leqslant \frac{\varepsilon}{2}$$

The sum S_2 is a bit more tricky. We use the identity $|a(x) - b(x)| \leqslant ||a(x)|| + ||b(x)||$.

$$\begin{aligned} S_2 &= \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) \\ &\leqslant 2||f|| \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} B_{n,k}(x) \\ &\leqslant 2||f|| \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} \frac{\left|\frac{k}{n} - x\right|^2}{\left|\frac{k}{n} - x\right|^2} B_{n,k}(x) \\ &\leqslant \frac{2||f||}{\delta^2} \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} \left|\frac{k}{n} - x\right|^2 B_{n,k}(x) \\ &\leqslant \left(\frac{2||f||}{\delta^2}\right) \left(\frac{x(1-x)}{n}\right) \end{aligned}$$

Note that $x(1-x) \leqslant \frac{1}{4}$ for $x \in [0,1]$. Thus we have $S_2 \leqslant \frac{||f||}{2\delta^2 n}$. The only part of the solution now remaining is to pick n such that $n > \frac{||f||}{\delta^2 \varepsilon}$. ■

(6) We are given a polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$ where $a_i \in \mathbb{Z}$. We are also given 4 integers $\alpha \neq \beta \neq \gamma \neq \delta$ such that $p(\alpha) = p(\beta) = p(\gamma) = p(\delta) = 7$. Thus we can rewrite p as

$$p(x) = 7 + (x - \alpha)(x - \beta)(x - \gamma)(x - \delta) q(x)$$

Substituting $p(s) = 10$ gives us $(s - \alpha)(s - \beta)(s - \gamma)(s - \delta) q(s) = 3$. Now note that since $q(x)$ has integer coefficients, $q(s)$ must take an integer value. Given distinct $\alpha, \beta, \gamma, \delta$, we must have distinct $(s - \alpha), (s - \beta), (s - \gamma), (s - \delta)$. Since 3 is a prime number, this is not possible (why? - take modulus of both sides). ■