ME 226 (Mechanical Measurements) - Homework 1

Name: Om Prabhu
Roll Number: 19D170018

Honor Pledge: I have abided by the course honor code. I have neither received solutions from my peers nor shared my solutions with them. All the solutions in this document are my own work.

---

(1) We have the data given as follows:

Input (MPa)	Increasing Output (MPa)	Decreasing Output (MPa)
0	0.25	0.2
10	10.56	10.6
20	21.65	21.75
30	32.21	32.65
40	43.65	43.98
50	52.3	52.73

i) According to least squares method, we have the following equations:

$$m = \frac{1}{D}\left(N\sum_k x_k y_k - \sum_k x_k \sum_k y_k\right)$$
$$c = \frac{1}{D}\left(\sum_k x_k^2 \sum_k y_k - \sum_k x_k \sum_k x_k y_k\right)$$
and $D = N\sum_k x_k^2 - \left(\sum_k x_k\right)^2$

The individual quantities can be calculated as shown below:

$$\sum_k x_k = 2(0+10+20+30+40+50) = 300$$
$$\sum_k x_k^2 = 2(0+100+400+900+1600+2500) = 11000$$
$$\sum_k x_k y_k = 105.6+106+433+435+966.3+979.5+1746+1759.2+2615+2636.5 = 11786.1$$
$$\sum_k y_k = 0.25+0.2+10.56+10.6+21.65+21.75+32.21+32.65+43.65+\ldots \ldots 43.98+52.3+52.73 = 322.63$$

Thus we can find the value of D, m and c as follows:

$$D = 12(11000) - (300)^2 = 42000$$
$$\boxed{\therefore m = \frac{12(11786.1) - (300)(322.63)}{42000} = 1.063}$$
$$\boxed{\therefore c = \frac{(11000)(322.63) - (300)(11786.1)}{42000} = 0.312}$$

Thus the equation of best linear fit for the given data is $y = 1.063x + 0.312$

ii) First, we find the standard deviation of y:

$$\sigma_y^2 = \frac{1}{N-2}\sum_k (mx_k + c - y_k)^2$$
$$= \frac{1}{10}\left[\begin{array}{l} 0.062^2 + 0.112^2 + 0.382^2 + 0.342^2 + (-0.078)^2 + (-0.178)^2 + \ldots \\ (-0.008)^2 + (-0.448)^2 + (-0.818)^2 + (-1.148)^2 + 1.162^2 + 0.732^2 \end{array}\right] = 0.4624$$

Thus we get $\boxed{\sigma_y = 0.68}$.

Based on this, we can calculate the remaining standard deviations as follows:

$$\therefore \sigma_x^2 = \frac{\sigma_y^2}{m^2} = 0.4092 \implies \boxed{\sigma_x = 0.64}$$

$$\therefore \sigma_m^2 = \frac{N\sigma_y^2}{D} = 1.321 \times 10^{-4} \implies \boxed{\sigma_m = 0.011}$$

$$\therefore \sigma_c^2 = \frac{\sigma_y^2 \sum_k x_k^2}{D} = 0.1211 \implies \boxed{\sigma_c = 0.35}$$

iii) We have the output value $y = 25.35$ and want to find the true input value. Using the best linear fit equation obtained earlier:

$$y = 1.063x + 0.312$$

$$\therefore x = \frac{y - 0.312}{1.063}$$

$$\therefore x = \frac{25.35 - 0.312}{1.063} = 23.55$$

The uncertainty of the input value within $\pm 3\sigma$ limits is $3 \times 0.64 = 1.92$. Thus the final value of the input is $\boxed{x = 23.55 \pm 1.92 \text{ MPa}}$.

(2) We have 15 sample values of L and D each and we know that surface area $A = \pi DL + \pi\frac{D^2}{2}$. We first calculate the value of surface area at each of the sample points:

$$
\begin{aligned}
A_1 &= \pi(20.4)(100.1) + \pi\frac{(20.4)^2}{2} = 7068.96 \\
A_2 &= \pi(19.5)(99.9) + \pi\frac{(19.5)^2}{2} = 6717.275 \\
A_3 &= \pi(20.1)(99.8) + \pi\frac{(20.1)^2}{2} = 6936.589 \\
A_4 &= \pi(19.9)(100.2) + \pi\frac{(19.9)^2}{2} = 6886.324 \\
A_5 &= \pi(20.1)(100.3) + \pi\frac{(20.1)^2}{2} = 6968.162 \\
A_6 &= \pi(20.3)(99.8) + \pi\frac{(20.3)^2}{2} = 7011.988 \\
A_7 &= \pi(20.2)(100) + \pi\frac{(20.2)^2}{2} = 6986.965 \\
A_8 &= \pi(19.6)(100.4) + \pi\frac{(19.6)^2}{2} = 6785.589 \\
A_9 &= \pi(19.7)(99.7) + \pi\frac{(19.7)^2}{2} = 6779.981 \\
A_{10} &= \pi(19.8)(99.9) + \pi\frac{(19.8)^2}{2} = 6829.948 \\
A_{11} &= \pi(20)(101) + \pi\frac{(20)^2}{2} = 6974.336 \\
A_{12} &= \pi(19.2)(99) + \pi\frac{(19.2)^2}{2} = 6550.598 \\
A_{13} &= \pi(19.4)(99.5) + \pi\frac{(19.4)^2}{2} = 6655.401 \\
A_{14} &= \pi(20)(100.6) + \pi\frac{(20)^2}{2} = 6949.203 \\
A_{15} &= \pi(20.4)(100.5) + \pi\frac{(20.4)^2}{2} = 7094.596
\end{aligned}
$$

From the calculated data, the mean value $\bar{A}$ is:

$$\bar{A} = \frac{\sum_{i=1}^{N} A_i}{N} = \frac{103195.915}{15} = 6879.728$$

From this data we can now calculate the sample variance as follows:

$$\sigma_A^2 = \sum_{i=1}^{N} \frac{(A_i - \bar{A})^2}{N-1} = 24441.0175 \implies \sigma_A = 156.336$$

Thus the final value of the area within $\pm 3\sigma$ error limits is $\boxed{A = 6879.728 \pm 469.008 \text{ mm}^2}$.

(3) Before we employ the least squares method to solve this problem, we need to prove the expressions for it first. We know that the input-output relation for the data is $q_0 = aq_i + b$. For convenience, we can rewrite this as $y = ax_b$, where y represents the output and x represents the input. We can then define the error E as:

$$E = \sum_{i=1}^{N} (y_i - (ax_i + b))^2$$

For minimum error, we must have $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$.

$$\frac{\partial E}{\partial a} = 0 \implies \sum_{i=1}^{N} 2(-x_i)(y_i - (ax_i + b)) = 0$$
$$\frac{\partial E}{\partial b} = 0 \implies \sum_{i=1}^{N} 2(y_i - (ax_i + b)) = 0$$

We can further simplify the above equations to get the following system of equations:

$$a\left(\sum_{i-1}^{N} x_i^2\right) + b\left(\sum_{i-1}^{N} x_i\right) = \left(\sum_{i-1}^{N} x_i y_i\right)$$
$$a\left(\sum_{i-1}^{N} x_i\right) + bN = \left(\sum_{i-1}^{N} y_i\right)$$

This system of linear equations can further be expressed in a matrix form as follows:

$$\begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix} \implies \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{D} \begin{bmatrix} N & -\sum_{i=1}^{N} x_i \\ -\sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix}$$

where the determinant $D = N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2$. Thus, the final expressions for a and b are:

$$a = \frac{1}{D}\left(N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right)$$
$$b = \frac{1}{D}\left(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i\right)$$

From the given data, we have $\sum_{i=1}^{N} x_i = 27$, $\sum_{i=1}^{N} x_i^2 = 179$, $\sum_{i=1}^{N} x_i y_i = 180.6$ and $\sum_{i=1}^{N} y_i = 27.35$.

Thus we can easily calculate the values of a, b, D as follows:

$$D = N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 = 7(179) - (27)^2 = 524$$

$$\therefore a = \frac{1}{D}\left(N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right) = \frac{1}{524}[7(180.6) - (27)(27.35)] = 1.003$$

$$\therefore b = \frac{1}{D}\left(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i\right) = \frac{1}{524}[(179)(27.35) - (180.6)(27)] = 0.037$$

Thus, $\boxed{a = 1.003}$ and $\boxed{b = 0.037}$. The linear best fit expression for the following set of data is $q_o = 1.003q_i + 0.037$. To calculate the accuracy of these results, we first need to find σ_y.

$$\begin{aligned}\sigma_y^2 &= \frac{1}{N-2}\sum_{i=1}^{N}(ax_i + b - y_i)^2 \\ &= \frac{1}{5}[(-0.063)^2 + (-0.06)^2 + (-0.007)^2 + (0.146)^2 + (-0.051)^2 + (0.158)^2 + (-0.133)^2] \\ &= 0.0148\end{aligned}$$

Thus we get $\boxed{\sigma_y = 0.122}$ and accuracy of measurement based on $\pm 3\sigma$ limits is ± 0.366. Now to find the accuracy of the calculated values of a and b:

$$\sigma_a^2 = \frac{N\sigma_y^2}{D} = 1.977 \times 10^{-4} \implies \boxed{\sigma_a = 0.014}$$

$$\sigma_b^2 = \frac{\sigma_y^2 \sum_i x_i^2}{D} = 5.056 \times 10^{-3} \implies \boxed{\sigma_b = 0.071}$$

(4) We have the relation $\kappa = q\dfrac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)}$. The root sum square error can be found as follows:

$$E_{RSS} = \sqrt{\left(\frac{\partial \kappa}{\partial q}\Delta q\right)^2 + \left(\frac{\partial \kappa}{\partial r_0}\Delta r_0\right)^2 + \left(\frac{\partial \kappa}{\partial r_1}\Delta r_1\right)^2 + \left(\frac{\partial \kappa}{\partial l}\Delta l\right)^2 + \left(\frac{\partial \kappa}{\partial T_1}\Delta T_1\right)^2 + \left(\frac{\partial \kappa}{\partial T_0}\Delta T_0\right)^2}$$

Let us first calculate the individual terms:

$$\frac{\partial \kappa}{\partial q}\Delta q = \left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)}\right)\Delta q = \frac{\ln 2}{2\pi(0.5)(30)} \times 1 = 7.35 \times 10^{-3}\ \text{W/m/K}$$

$$\frac{\partial \kappa}{\partial r_0}\Delta r_0 = \left(\frac{1}{r_0}\right)\frac{q}{2\pi l(T_1 - T_0)}\Delta r_0 = \frac{70}{2\pi(0.02)(0.5)(30)} \times 0.0005 = 18.57 \times 10^{-3}\ \text{W/m/K}$$

$$\frac{\partial \kappa}{\partial r_1}\Delta r_1 = -\left(\frac{1}{r_1}\right)\frac{q}{2\pi l(T_1 - T_0)}\Delta r_1 = -\frac{70}{2\pi(0.01)(0.5)(30)} \times 0.0005 = -37.13 \times 10^{-3}\ \text{W/m/K}$$

$$\frac{\partial \kappa}{\partial l}\Delta l = -q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l^2(T_1 - T_0)}\right)\Delta l = -\frac{70\ln 2}{2\pi(0.5)^2(30)} \times 0.01 = -10.29 \times 10^{-3}\ \text{W/m/K}$$

$$\frac{\partial \kappa}{\partial T_1}\Delta T_1 = -q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)^2}\right)\Delta T_1 = -\frac{70\ln 2}{2\pi(0.5)(30)^2} \times 1 = -17.15 \times 10^{-3}\ \text{W/m/K}$$

$$\frac{\partial \kappa}{\partial T_0}\Delta T_0 = q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)^2}\right)\Delta T_0 = \frac{70\ln 2}{2\pi(0.5)(30)^2} \times 1 = 17.15 \times 10^{-3}\ \text{W/m/K}$$

Thus we have, $E_{RSS} = \sqrt{(7.35)^2 + (18.57)^2 + (-37.13)^2 + (-10.29)^2 + (-17.15)^2 + (17.15)^2} \times 10^{-3}$ i.e. $\boxed{E_{RSS} = 4.965 \times 10^{-2}\ \text{W/m/K}}$.

The maximum possible error in measurement of thermal conductivity is simply the absolute sum error.

$$E_{max} = \left|\frac{\partial \kappa}{\partial q}\Delta q\right| + \left|\frac{\partial \kappa}{\partial r_0}\Delta r_0\right| + \left|\frac{\partial \kappa}{\partial r_1}\Delta r_1\right| + \left|\frac{\partial \kappa}{\partial l}\Delta l\right| + \left|\frac{\partial \kappa}{\partial T_1}\Delta T_1\right| + \left|\frac{\partial \kappa}{\partial T_0}\Delta T_0\right|$$

Thus the maximum possible error is $\boxed{E_{max} = 1.076 \times 10^{-1}\ \text{W/m/K}}$.

(5) We have the relation $F = \dfrac{\epsilon E b t^2}{3r(\sin\theta - \frac{2}{\pi})}$. We are also given that the root sum square limit of the load is 2.5 kN. We also know that

$$E_{RSS}^2 = \left(\frac{\partial\kappa}{\partial q}\Delta q\right)^2 + \left(\frac{\partial\kappa}{\partial r_0}\Delta r_0\right)^2 + \left(\frac{\partial\kappa}{\partial r_1}\Delta r_1\right)^2 + \left(\frac{\partial\kappa}{\partial l}\Delta l\right)^2 + \left(\frac{\partial\kappa}{\partial T_1}\Delta T_1\right)^2 + \left(\frac{\partial\kappa}{\partial T_0}\Delta T_0\right)^2$$

All of the variances are unknown to us. We ignore the error in E (given in the question). However, we can calculate the values of the derivatives:

$$\frac{\partial F}{\partial \epsilon} = \frac{Ebt^2}{3r(\sin\theta - \frac{2}{\pi})} = \frac{(210\times 10^9)(25\times 10^{-3})(36\times 10^{-6})}{3(75\times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.19\times 10^7$$

$$\frac{\partial F}{\partial b} = \frac{\epsilon E t^2}{3r(\sin\theta - \frac{2}{\pi})} = \frac{(210\times 10^9)(3400\times 10^{-6})(36\times 10^{-6})}{3(75\times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.62\times 10^6$$

$$\frac{\partial F}{\partial t} = \frac{2\epsilon E b t}{3r(\sin\theta - \frac{2}{\pi})} = \frac{2(3400\times 10^{-6})(210\times 10^9)(25\times 10^{-3})(6\times 10^{-3})}{3(75\times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.35\times 10^7$$

$$\frac{\partial F}{\partial r} = -\frac{\epsilon E b t^2}{3r^2(\sin\theta - \frac{2}{\pi})} = -\frac{(3400\times 10^{-6})(210\times 10^9)(25\times 10^{-3})(36\times 10^{-6})}{3(75\times 10^{-3})^2(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = -5.41\times 10^5$$

$$\frac{\partial F}{\partial \theta} = -\frac{\epsilon E b t^2\cos\theta}{3r(\sin\theta - \frac{2}{\pi})^2} = -\frac{(3400\times 10^{-6})(210\times 10^9)(25\times 10^{-3})(36\times 10^{-6})(\frac{1}{\sqrt{2}})}{3(75\times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})^2} = 4.07\times 10^5$$

By method of equal effects, each of the quantities on RHS will be $\dfrac{2500^2}{5} = 1.25\times 10^6$ N^2. Thus we have:

$$\left(\frac{\partial F}{\partial \epsilon}\right)^2 \Delta\epsilon^2 = 1.25\times 10^6 \implies \Delta\epsilon = 9.39\times 10^{-5} \implies \boxed{\sigma_\epsilon = 3.13\times 10^{-5}}$$

$$\left(\frac{\partial F}{\partial b}\right)^2 \Delta b^2 = 1.25\times 10^6 \implies \Delta b = 6.90\times 10^{-4} \implies \boxed{\sigma_b = 2.30\times 10^{-4}}$$

$$\left(\frac{\partial F}{\partial t}\right)^2 \Delta t^2 = 1.25\times 10^6 \implies \Delta t = 8.28\times 10^{-5} \implies \boxed{\sigma_t = 2.76\times 10^{-5}}$$

$$\left(\frac{\partial F}{\partial r}\right)^2 \Delta r^2 = 1.25\times 10^6 \implies \Delta r = 2.07\times 10^{-3} \implies \boxed{\sigma_r = 6.90\times 10^{-4}}$$

$$\left(\frac{\partial F}{\partial \theta}\right)^2 \Delta\theta^2 = 1.25\times 10^6 \implies \Delta\theta = 2.75\times 10^{-3} \implies \boxed{\sigma_\theta = 9.17\times 10^{-4}}$$

If uncertainty is based on absolute limits, then each of the quantities on RHS is $\dfrac{2500}{5} = 500$ N.

$$\left|\frac{\partial F}{\partial \epsilon}\right| \Delta\epsilon = 500 \implies \Delta\epsilon = 4.20\times 10^{-5} \implies \boxed{\sigma_\epsilon = 1.40\times 10^{-5}}$$

$$\left|\frac{\partial F}{\partial b}\right| \Delta b = 500 \implies \Delta b = 3.09\times 10^{-4} \implies \boxed{\sigma_b = 1.03\times 10^{-4}}$$

$$\left|\frac{\partial F}{\partial t}\right| \Delta t = 500 \implies \Delta t = 3.70\times 10^{-5} \implies \boxed{\sigma_t = 1.23\times 10^{-5}}$$

$$\left|\frac{\partial F}{\partial r}\right| \Delta r = 500 \implies \Delta r = 9.24\times 10^{-4} \implies \boxed{\sigma_r = 3.08\times 10^{-4}}$$

$$\left|\frac{\partial F}{\partial \theta}\right| \Delta\theta = 500 \implies \Delta\theta = 1.23\times 10^{-3} \implies \boxed{\sigma_\theta = 4.21\times 10^{-4}}$$