ME 226 - Mechanical Measurements

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Note To Reader

This document is a compilation of the notes I made while taking the course ME 226 (Mechanical Measurements) in my 4$^{\text{th}}$ semester at IIT Bombay. It is not a substitute for any formal lecture or textbook on the subject, since I pretty much overlook all the theory parts.

If you have any suggestions and/or spot any errors, you know where to contact me.

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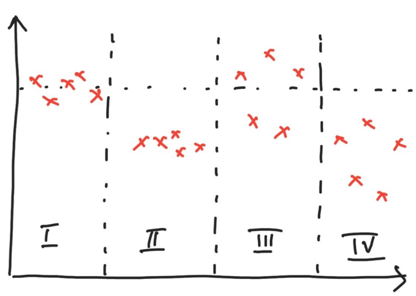
1 Introduction & Basic Concepts

Some definitions:

– sensitivity: slope of the output vs input curve for an instrument
– span: difference between maximum and minimum possible measurements for an instrument
– range: difference between maximum and minimum deflection for an instrument
– resolution: smallest measurable change in input
– threshold: smallest measurable input
– hysteresis: inability of instrument to give repeatable results during loading and unloading (hysteresis loss = area under the input-output curve)

Error in an instrument is a combination of 2 factors - bias (correctable by calibration) and imprecision (permanent component caused due to human error).



I – no bias, no imprecision

II – bias, no imprecision

III – no bias, imprecision

IV – bias, imprecision

Additionally, results should be fairly repeatable (i.e. repeating the measurements should yield similar values).

Basic Statistics:

$$\text{probability density function} = \frac{\text{(number of readings in an interval)}}{\text{(total number of readings)} \times \text{(width of interval)}}$$

Plot pdf as a function of interval length – area under the curve is 1. On dividing the data into very small intervals, the pdf is a continuous function $f(x)$ such that $P(a < x < b) = \displaystyle\int_a^b f(x)\mathrm{d}x$.

In practice, many measurement sets are very close to the Gaussian distribution $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. For an ideal condition $-\infty < x < \infty$, but instruments cannot have infinite range.

	for a population	for a sample
mean value	$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$	$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$
variance	$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}$	$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}$

A population refers to a continuous data distribution whereas a sample refers to the fixed number of discrete data points. 68%, 95%, 99.7% readings lie in the $\pm\sigma, \pm2\sigma, \pm3\sigma$ range respectively.

Method of Least Squares:

Assume a linear fit $y = mx + c$. We define the error $E = \sum_{k=1}^{N} ((mx_k + c) - y_k)^2$. In order to minimize the error,

$$\frac{\partial E}{\partial m} = 0 \implies \sum 2(mx_k + c - y_k)x_k = 0$$

$$\frac{\partial E}{\partial c} = 0 \implies \sum 2(mx_k + c - y_k) = 0$$

Solving this as a system of linear equations, we get

$$m = \frac{1}{D}\left(N \sum x_k y_k - \sum x_k \sum y_k\right)$$

$$c = \frac{1}{D}\left(N \sum x_k^2 \sum y_k - \sum x_k \sum x_k y_k\right)$$

$$D = N \sum x_k^2 - \left(\sum x_k\right)^2$$

The variances in y, x, m, c are calculated using the following formulae:

$$s_y^2 = \frac{1}{N-2} \sum (mx_k + c - y_k)^2 \qquad s_x^2 = \frac{s_y^2}{m^2}$$

$$s_m^2 = \frac{N s_y^2}{N \sum x_k^2 - (\sum x_k)^2} \qquad s_c^2 = \frac{s_y^2 \sum x_k^2}{N \sum x_k^2 - (\sum x_k)^2}$$

The Error Function:

We have the Gaussian distribution given by $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Define $\eta = \frac{x-\mu}{\sigma\sqrt{2}}$. The error function is defined as $\mathrm{er}f(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-t^2} \mathrm{d}t$. It follows that $\mathrm{er}f(-\eta) = -\mathrm{er}f(\eta)$.

$$P(X < x) = F(x) = \frac{1}{2}(1 + \mathrm{er}f(\eta))$$

$$P(x_1 < X < x_2) = F(x_2) - F(x_1) = \frac{1}{2}(\mathrm{er}f(\eta_2) - \mathrm{er}f(\eta_1))$$

A table for error function values is as follows:

x	Hundredths digit of x 0	1	2	3	4	5	6	7	8	9
0.0	0.00000	0.01128	0.02256	0.03384	0.04511	0.05637	0.06762	0.07886	0.09008	0.10128
0.1	0.11246	0.12362	0.13476	0.14587	0.15695	0.16800	0.17901	0.18999	0.20094	0.21184
0.2	0.22270	0.23352	0.24430	0.25502	0.26570	0.27633	0.28690	0.29742	0.30788	0.31828
0.3	0.32863	0.33891	0.34913	0.35928	0.36936	0.37938	0.38933	0.39921	0.40901	0.41874
0.4	0.42839	0.43797	0.44747	0.45689	0.46623	0.47548	0.48466	0.49375	0.50275	0.51167
0.5	0.52050	0.52924	0.53790	0.54646	0.55494	0.56332	0.57162	0.57982	0.58792	0.59594
0.6	0.60386	0.61168	0.61941	0.62705	0.63459	0.64203	0.64938	0.65663	0.66378	0.67084
0.7	0.67780	0.68467	0.69143	0.69810	0.70468	0.71116	0.71754	0.72382	0.73001	0.73610
0.8	0.74210	0.74800	0.75381	0.75952	0.76514	0.77067	0.77610	0.78144	0.78669	0.79184
0.9	0.79691	0.80188	0.80677	0.81156	0.81627	0.82089	0.82542	0.82987	0.83423	0.83851
1.0	0.84270	0.84681	0.85084	0.85478	0.85865	0.86244	0.86614	0.86977	0.87333	0.87680
1.1	0.88021	0.88353	0.88679	0.88997	0.89308	0.89612	0.89910	0.90200	0.90484	0.90761
1.2	0.91031	0.91296	0.91553	0.91805	0.92051	0.92290	0.92524	0.92751	0.92973	0.93190
1.3	0.93401	0.93606	0.93807	0.94002	0.94191	0.94376	0.94556	0.94731	0.94902	0.95067
1.4	0.95229	0.95385	0.95538	0.95686	0.95830	0.95970	0.96105	0.96237	0.96365	0.96490
1.5	0.96611	0.96728	0.96841	0.96952	0.97059	0.97162	0.97263	0.97360	0.97455	0.97546
1.6	0.97635	0.97721	0.97804	0.97884	0.97962	0.98038	0.98110	0.98181	0.98249	0.98315
1.7	0.98379	0.98441	0.98500	0.98558	0.98613	0.98667	0.98719	0.98769	0.98817	0.98864
1.8	0.98909	0.98952	0.98994	0.99035	0.99074	0.99111	0.99147	0.99182	0.99216	0.99248
1.9	0.99279	0.99309	0.99338	0.99366	0.99392	0.99418	0.99443	0.99466	0.99489	0.99511
2.0	0.99532	0.99552	0.99572	0.99591	0.99609	0.99626	0.99642	0.99658	0.99673	0.99688
2.1	0.99702	0.99715	0.99728	0.99741	0.99753	0.99764	0.99775	0.99785	0.99795	0.99805
2.2	0.99814	0.99822	0.99831	0.99839	0.99846	0.99854	0.99861	0.99867	0.99874	0.99880
2.3	0.99886	0.99891	0.99897	0.99902	0.99906	0.99911	0.99915	0.99920	0.99924	0.99928
2.4	0.99931	0.99935	0.99938	0.99941	0.99944	0.99947	0.99950	0.99952	0.99955	0.99957
2.5	0.99959	0.99961	0.99963	0.99965	0.99967	0.99969	0.99971	0.99972	0.99974	0.99975
2.6	0.99976	0.99978	0.99979	0.99980	0.99981	0.99982	0.99983	0.99984	0.99985	0.99986
2.7	0.99987	0.99987	0.99988	0.99989	0.99989	0.99990	0.99991	0.99991	0.99992	0.99992
2.8	0.99992	0.99993	0.99993	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995	0.99996
2.9	0.99996	0.99996	0.99996	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998
3.0	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999	0.99999
3.1	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
3.2	0.99999	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Combination of Component Errors:

Measured quantities are often influenced by a combination of other measured quantities (for example, stored potential energy = $\rho g h$). Let quantity $P = f(u_1, u_2, \ldots, u_n)$ with individual errors $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$.

$$\text{absolute error} = \Delta P = \left|\frac{\partial f}{\partial u_1}\Delta u_1\right| + \left|\frac{\partial f}{\partial u_2}\Delta u_2\right| + \cdots + \left|\frac{\partial f}{\partial u_n}\Delta u_n\right|$$

$$\text{root sum square error} = E_{RSS} = \sqrt{\left(\frac{\partial f}{\partial u_1}\Delta u_1\right)^2 + \left(\frac{\partial f}{\partial u_2}\Delta u_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial u_n}\Delta u_n\right)^2}$$

For N measurements of each of the quantities,

$$\sigma_P^2 = \left(\frac{\partial f}{\partial u_1}\right)\sigma_1^2 + \left(\frac{\partial f}{\partial u_2}\right)\sigma_2^2 + \cdots + \left(\frac{\partial f}{\partial u_n}\right)\sigma_n^2$$

Error Analysis of Voltmeters and Ammeters:

For a voltmeter, we first calculate the equivalent resistance R_{eq} across the points where the voltmeter is to be connected. Then,

$$\text{measured voltage } E_m = \frac{R_m}{R_m + R_{eq}} E_0 \quad \text{and} \quad \text{error } \epsilon = 1 - \frac{E_m}{E_0} = \frac{R_{eq}}{R_m + R_{eq}}$$

For an ammeter, we again calculate R_{eq} (this time the meter will be in series with the rest of the circuit). Then,

$$\text{measured current } I_m = \frac{R_{eq}}{R_m + R_{eq}} I_u \quad \text{and} \quad \text{error } \epsilon = 1 - \frac{I_m}{I_u} = \frac{R_m}{R_m + R_{eq}}$$

2 Dynamic Characteristics

General mathematical model (input $q_i \rightarrow$ output q_0) of a system can be represented by:

$$a_n \frac{\mathrm{d}^n q_0}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} q_0}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}q_0}{\mathrm{d}t} + a_0 q_0 = b_m \frac{\mathrm{d}^m q_i}{\mathrm{d}t^m} + b_{m-1} \frac{\mathrm{d}^{m-1} q_i}{\mathrm{d}t^{m-1}} + \cdots + b_1 \frac{\mathrm{d}q_i}{\mathrm{d}t} + b_0 q_i$$

Normally we don't specify the input derivatives, so we replace the RHS by just q_i. Sometimes we may also need to employ techniques like the Laplace transform to solve certain problems.

$\int_0^t f(t-\tau)\, g(\tau)\, d\tau$	$F(s)\, G(s)$
$f(t+T) = f(t)$	$\dfrac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Zero Order Systems:

The general equation can be written as $a_0 q_0 = b_0 q_i \implies q_0 = \dfrac{b_0}{a_0} q_i = K q_i$.

- K is the static sensitivity of the system
- output is instantaneous with respect to input (i.e. $\phi = 0$)

An example of a zero order system is a potentiometer. The emf $e_0 = \dfrac{x}{L} E_b$ is a function of only variable, i.e. distance of the sliding contact.

2.1 First Order Systems

The general equation characterizing a first order system is:

$$a_1 \frac{\mathrm{d}q_0}{\mathrm{d}t} + a_0 q_0 = b_0 q_i$$
$$\therefore \frac{a_1}{a_0}\frac{\mathrm{d}q_0}{\mathrm{d}t} + q_0 = \frac{b_0}{a_0} q_i$$
$$\therefore (\tau D + 1) q_0 = K q_i \implies \frac{q_0}{q_i}(D) = \frac{K}{1 + \tau D}$$

τ is the time constant whereas K is the static sensitivity of the system.

With certain assumptions, we can model a thermometer as a 1st order system. Its relies on thermal expansion of the liquid column in response to changes in the surrounding temperature.

$$\left.\begin{aligned} \beta &= \text{coefficient of volume expansion} \\ V &= \text{volume of bulb} \\ A_c &= \text{cross-sectional area of capillary} \\ \rho &= \text{density of thermometer fluid} \\ C_p &= \text{specific heat capacity of thermometer fluid} \\ h &= \text{heat transfer coefficient} \\ A_s &= \text{surface area of bulb} \end{aligned}\right\} K = \frac{\beta V}{A_c} \quad \text{and} \quad \tau = \frac{\rho C_p V}{h A_s}$$

The differential equation obtained is:

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{hA_s}{\rho C_p V} T = \frac{hA_s T_f}{\rho C_p V} \implies \frac{\mathrm{d}y}{\mathrm{d}t} + p(t) y = g(t)$$

$$y(t) = \frac{\int e^{\int p(t)\mathrm{d}t} g(t)\mathrm{d}t + C}{e^{\int p(t)\mathrm{d}t}} \implies T = T_f + (T_0 - T_f)e^{-t/\tau}$$

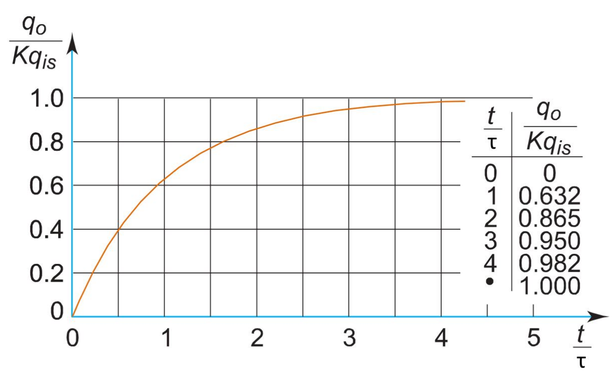
Step Response:

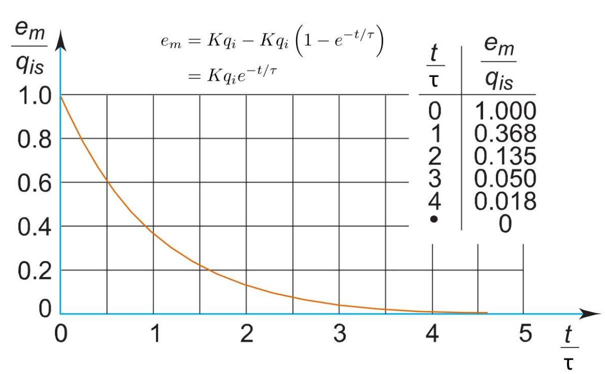
For a step response, the input q_i is constant. Hence, the governing equation is:

$$(\tau D + 1) q_0 = K q_i \implies q_0 = K q_i + C e^{-t/\tau})$$

For zero initial conditions, we have $q_0 = K q_i (1 - e^{-t/\tau})$. Thus, the response time depends only on the value of τ. The error for a step response can be written as

$$e_m = K q_i - q_0 = K q_i e^{-t/\tau}$$





Ramp Response:

The governing equation is $(\tau D+1)q_0 = Kq_{iramp}t$. Applying the Laplace transform, we get

$$q_0 = \frac{Kq_{iramp}t}{(1+\tau D)} \implies Q_0(s) = \frac{Kq_{iramp}}{s^2(1+\tau s)} \implies \frac{Q_0(s)}{Kq_{iramp}} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{1}{s+\frac{1}{\tau}}$$

Inverting the Laplace transform, we finally get

$$q_0(t) = Kq_{iramp}(t-\tau) + Kq_{iramp}\tau e^{-t/\tau} \quad \text{and} \quad e_m = Kq_{iramp}\tau(1-e^{-t/\tau})$$

The steady state error (i.e. component of error that stays constant with time) is given by $e_{ss} = Kq_{iramp}\tau$ (often we assume $K=1$). The transient error eventually converges to 0, thus there is always an error of e_{ss} even for very large values of time.

Impulse Response:

We initially assume a step input of magnitude A/T applied for time T. The impulse response can then be found in the limit $T \to 0$.

$$q_0 = \frac{A}{T}(1-e^{-t/\tau}) \text{ for } 0 \leqslant t \leqslant T$$

$$q_0 = \frac{A(1-e^{-T/\tau})e^{-t/\tau}}{Te^{-T/\tau}} \text{ for } t > T$$

In the limit $T \to 0$, we get the impulse response as

$$q_0 = \frac{A}{T}e^{-t/\tau}$$

Frequency Response:

$$\frac{q_0}{Kq_i} = \frac{1}{1+\tau D} = \frac{1}{1+j\tau\omega} \implies \frac{q_0}{Kq_i} = \frac{1}{\sqrt{1+\tau^2\omega^2}};\ \phi = \arctan(-\tau\omega)$$

$$\therefore \text{ for input } q_i = a\sin(\omega t) \to \text{output } q_0 = \frac{a}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t+\phi)$$

As observed, the frequency response has a magnitude as well as a phase difference associated with it. An ideal frequency response would have $\frac{q_0}{Kq_i} = 1$ and $\phi = 0$.

2.2 Second Order Systems

The general equation characterizing a second order system is:

$$a_2\frac{\mathrm{d}^2q_0}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}q_0}{\mathrm{d}t} + a_0q_0 = b_0q_i$$

$$\therefore \frac{a_2}{a_0}\frac{\mathrm{d}^2q_0}{\mathrm{d}t^2} + \frac{a_1}{a_0}\frac{\mathrm{d}q_0}{\mathrm{d}t} + q_0 = \frac{b_0}{a_0}q_i$$

A very common example of a second order system is that of a mass, spring and damper. The force applied by the spring depends on the displacement x while the force applied by the damper depends on the velocity v.

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F - Kx - B\frac{\mathrm{d}x}{\mathrm{d}t} \implies (mD^2 + BD + K)x = F \implies \left(\frac{m}{k}D^2 + \frac{B}{K}D + 1\right)x = \frac{F}{K}$$

Replacing $\omega_n = \sqrt{\frac{K}{m}}$ and $\zeta = \frac{B}{2\sqrt{mK}}$, we get

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right)x = \frac{F}{k}$$

Step, Ramp & Impulse Responses:

All of the following equations can be derived using the fundamental differential equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$$

Damping ratio	Input $f(t)$	Characteristic Response $y(t)$
$0 \leq \zeta < 1$	$f(t) = u_r(t)$	$y_r(t) = \frac{1}{\omega_n^2}\left[t + \frac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t + \frac{2\zeta^2 - 1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) - \frac{2\zeta}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi)\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)$
$\zeta = 1$	$f(t) = u_r(t)$	$y_r(t) = \frac{1}{\omega_n^2}\left[t + \frac{2}{\omega_n}e^{-\omega_n t} + te^{-\omega_n t} - \frac{2}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \frac{1}{\omega_n^2}\left[1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t}\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = te^{-\omega_n t}$
$\zeta > 1$	$f(t) = u_r(t)$	$y_r(t) = \frac{1}{\omega_n^2}\left[t + \frac{\omega_n}{2\sqrt{1-\zeta^2}}\left(\tau_1^2 e^{-t/\tau_1} - \tau_2^2 e^{-t/\tau_2}\right) - \frac{2\zeta}{\omega_n}\right]$
	$f(t) = u_s(t)$	$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = \frac{1}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$

- damped natural frequency $\omega_d = \sqrt{1-\zeta^2}\omega_n$ for $0 \leqslant \zeta < 1$
- phase angle $\psi = \arctan(\frac{\zeta}{\sqrt{1-\zeta^2}})$ for $0 \leqslant \zeta < 1$
- time constants for overdamped ($\zeta > 1$) systems are

$$\tau_1 = \frac{1}{\zeta\omega_n - \sqrt{\zeta^2-1}\omega_n} \text{ and } \tau_2 = \frac{1}{\zeta\omega_n + \sqrt{\zeta^2-1}\omega_n}$$

The impulse response can be found by simply differentiating the step response. Similarly, the ramp response can be found by integrating the step response.

For a ramp response, the steady state error is given by

$$e_{m,ss} = \frac{2\zeta q_{iramp}}{\omega_n}$$

Some important observations from the above equations are as follows:

- overdamped systems have a sluggish response (i.e. large time delay to reach desired output)
- underdamped system have an oscillatory response depending on the damping coefficient
- critically damped systems have the most desirable performance
- in most systems, $\omega_n t$ is determined by the response, so we often try to design ω_n to be as large as possible
- most commercial systems tend to use $0.6 < \zeta < 0.7$, since the system gives $\approx 90\%$ accuracy at $\omega_n t = 2.5$

Frequency Response:

The general equation for frequency response of a second order system is:

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right) q_0 = K q_i \implies \frac{q_0}{K q_i} = \frac{1}{\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1} \implies \frac{q_0}{K q_i} = \frac{1}{\frac{-\omega^2}{\omega_n^2} + \frac{2\zeta\omega}{\omega_n} j + 1}$$

$$\therefore \frac{q_0}{K q_i} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}; \ \phi = \arctan\left[\frac{-2\zeta}{\frac{\omega_n}{\omega} - \frac{\omega}{\omega_n}}\right]$$

When ω/ω_n is small, the response for $0.6 < \zeta < 0.7$ is satisfactory. Also when the system frequency matches the natural frequency of the device, resonance occurs in which $\phi = 0$ and the amplitude rises.

2.3 Combination of Systems

For systems in series, their individual transfer functions are simply multiplied. For example, 2 first order systems in series give a second order system as follows:

$$\text{for } q_i = (\tau_1 D + 1) q_{01} \text{ and } q_{01} = (\tau_2 D + 1) q_0$$

$$q_i = (\tau_1 D + 1)(\tau_2 D + 1) q_0 \implies q_i = (\tau_1 \tau_2 D^2 + \tau_1 D + \tau_2 D + 1) q_o$$

Comparing this with the standard equation for a second order system, we get $\tau_1 \tau_2 = \frac{1}{\omega_n^2}$ and $\tau_1 + \tau_1 = \frac{2\zeta}{\omega_n}$.